MSc course 191551150 "Numerical techniques for PDEs"
Final test, January 19, 2017, 08:45–10:45

The parts A and B may be graded separately. Therefore please write your answers for the A and B parts on separate sheets of paper, indicating your name and student number on both A and B parts. The use of calculators and other electronic devices is not allowed. Motivate all your answers.

Part A. Parabolic PDEs Consider the following PDE

$$u_t - (Du_x)_x = 0, \tag{1}$$

with initial and boundary conditions given, where $u = u(x,t)$ is unknown and $D = D(x) > 0$ is given.

1pt **A1** Give a definition of a convergent numerical scheme for solving equation (1).

2pt **A2** Consider the following scheme to solve (1):

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{D_{j+1/2}(U_{j+1}^n - U_j^n) - D_{j-1/2}(U_j^n - U_{j-1}^n)}{(\Delta x)^2} = 0.$$

Define all symbols appearing in this expression. Moreover, carry out the practical check of the maximum principle (recall 'positivity' and 'sum property') to see whether the principle holds for some Δt.

3pt **A3** Assume that for the truncation error $T(x,t)$ of this numerical scheme holds

$$|T(x,t)| \leqslant C\Delta t,$$

where C is a constant. First, define the local error e_j^n of the above scheme and establish the discrete equation satisfied by it. Second, define the global error E^n of the numerical scheme solving equation (1) and derive an upper-bound for it. Third, show that E^n decreases as Δt decreases while C is kept fixed.

Part B. Hyperbolic PDEs

1pt **B1** We solve a partial differential equation

$$u_t + au_x = 0, \tag{2}$$

where $u = u(x,t)$ is unknown function and $a = a(x)$ is given. We have $0 < x < 1$ and $t > 0$ and there are also some given boundary conditions (which we do not consider in this question) and initial condition $u(x,0) = u^0(x)$, with given $u^0(x)$. For internal mesh points (away from the boundaries), formulate the first order upwind scheme for solving (2). Take into account that $a(x)$ may change sign (i.e., be positive for certain x and negative for some other x).

See the other side

2pt **B2** We now consider the same PDE and the same initial and boundary conditions as given in the previous question. Again, we neglect boundary conditions in this question. Assume that $a(x) > 0$ for all x and consider the following numerical scheme for solving this problem:

$$\frac{\delta_t(U_j^{n+1/2} + U_{j+1}^{n+1/2})}{2\Delta t} + a_{j+1/2}\frac{\delta_x(U_{j+1/2}^n + U_{j+1/2}^{n+1})}{2\Delta x} = 0, \tag{3}$$

where $\delta_t U_j^{n+1/2} = U_j^{n+1} - U_j^n$ and similar relations hold for $\delta_t U_{j+1}^{n+1/2}$, $\delta_x U_{j+1/2}^n$ and $\delta_x U_{j+1/2}^{n+1}$. Sketch the domain of dependence for this scheme and, based on it, derive a CFL condition for this scheme. Motivate your answer.

3pt **B3** For the same problem as in Question B1 and again neglecting boundary conditions, we consider numerical scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + (1-\theta)a_j\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \theta a_j\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = 0, \tag{4}$$

where $\theta \in [0,1]$. Assume that a is constant. Substitute a numerical Fourier mode $U_j^n = \lambda^n e^{ikj\Delta x}$ into the scheme to derive an expression for λ. In the formula to be obtained by you, λ should depend on θ and on the familiar parameters ξ and ν. We now consider two choices: $\theta = \frac{1}{2}$ and $\theta = 1$. For which ν does each of these choices give a stable scheme? What is the damping error for each choice of θ? Based on this analysis which value for θ, $\frac{1}{2}$ or 1, would you prefer? Motivate your answer.

The grade for the test is determined as $G = 1 + 9P/12$ where P is the number of points earned.