Mastermath, course Applied Finite Elements
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$$\text{Exam Grade} = \frac{\text{Sum over all credits}}{2}.$$

NB. This exam contains three questions!

1 Let Ω be a non-empty region in $\mathbb{R}^d$. We consider the following weak form:

$$(W): \text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \nabla u \cdot \nabla \phi d\Omega = \int_\Omega \phi f d\Omega, \qquad \forall \phi \in H_0^1(\Omega).$$

where $H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$, and $H^1(\Omega) := \{u \in L^2(\Omega) : \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)\}$. Let $\Sigma_h(\Omega)$ be a finite dimensional subspace of $H_0^1(\Omega)$. We search the finite element approximation of u in $\Sigma_h(\Omega)$, given by

$$(W_h): \text{Find } u_h \in \Sigma_h(\Omega) \text{ such that } \int_\Omega \nabla u_h \cdot \nabla \phi_h d\Omega = \int_\Omega \phi_h f d\Omega, \qquad \forall \phi_h \in \Sigma_h(\Omega).$$

a Let $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$, and let $||.||$ represent the vector norm, show that

$$\int_\Omega ||\nabla u||^2 d\Omega = 0 \Longrightarrow u = 0, \text{ in } \Omega.$$

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(1 pt)

b Use the form (W_h) and the weak form (W) to show that

$$\int_\Omega \nabla(u - u_h) \cdot \nabla \phi_h d\Omega = 0, \qquad \phi_h \in \Sigma_h(\Omega). \tag{1}$$

(1 pt) 8

c Let $\hat{u}_h \in \Sigma_h(\Omega)$, show that

$$0 \leq \int_\Omega ||\nabla(u - u_h)||^2 d\Omega = \int_\Omega \nabla(u - u_h) \cdot \nabla(u - \hat{u}_h) d\Omega, \qquad \hat{u}_h \in \Sigma_h(\Omega). \tag{2}$$ 8

Hint: Use the orthogonality relation. (1 pt)

d Show that the above relation (2) implies

$$0 \leq \int_\Omega ||\nabla(u - u_h)||^2 d\Omega \leq \int_\Omega ||\nabla(u - \hat{u}_h)||^2 d\Omega, \qquad \hat{u}_h \in \Sigma_h(\Omega). \tag{3}$$

(1 pt)

e Let $u_I \in \Sigma_h(\Omega)$ represent the interpolation of u onto $\Sigma_h(\Omega)$, further let h be a characteristic element diameter and p be the order of interpolation, then it is possible to prove that there exists a $K > 0$ such that

$$\left(\int_\Omega ||\nabla(u - u_I)||^2 d\Omega \right)^{\frac{1}{2}} \leq K h^p. \tag{4}$$

The above norm is referred to as the *energy norm* (in this case of the interpolatory error). Derive an upper bound for the energy norm of the finite element error under use of p-th order interpolatory basis functions. (1 pt)

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{a}/\left(d-\frac{bc}{a}\right) & \frac{1}{d-\frac{bc}{a}} \end{array}\right)$$

$a \cdot \det$

2 We use linear, triangular elements to solve a boundary value problem. We use an isoparametric transformation to derive the finite element method. We consider a triangular element e_k in physical space, $e_k \subset \Omega$, and this element has vertices with coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) in the x, y coordinate system. This element is mapped onto a reference element e, which is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the s, t coordinate system.

a Motivate that the transformation

$$T: e \to e_k, \quad T: \begin{cases} x = x(s,t) = x_1(1-s-t) + x_2 s + x_3 t, \\ y = y(s,t) = y_1(1-s-t) + y_2 s + y_3 t, \end{cases} \tag{5}$$

defines a linear (affine) transformation from e to e_k for $s, t \in [0,1]$. (1 pt)

b For the basis functions in the reference element e, we use the basis functions $\phi_1(s,t) = 1-s-t$, $\phi_2(s,t) = s$, $\phi_3(s,t) = t$. Let $(s_1,t_1) = (0,0)$, $(s_2,t_2) = (1,0)$ and $(s_3,t_3) = (0,1)$. Further, δ_{ij} represents the Kronecker Delta. Show that these basis functions satisfy $\phi_i(s_j,t_j) = \delta_{ij}$. (1 pt)

c Express the Jacobian matrix $\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$, and its determinant Δ in terms of the vertex coordinates of e_k. (1 pt)

d Calculate the Jacobian matrix $\frac{\partial(s,t)}{\partial(x,y)}$. Express your results in terms of the coordinates of the vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3) and the determinant Δ (which is the determinant from assignment 2c). (1 pt)

e Express $\frac{\partial \phi_1}{\partial x}$ in terms of Δ and the coordinate positions of the vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3). (1 pt)

f Compute $S_{11}^{e_k} = \int_{e_k} ||\nabla\phi_1||^2 d\Omega$ in terms of Δ and the coordinate positions of the vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3). (2 pt)

3 We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ (bounded by $\partial\Omega$):

$$\begin{cases} -\nabla\cdot(D(x,y)\nabla u) + \nabla\cdot(\mathbf{q}(x,y)u) + u = f(x,y), & \text{in } \Omega, \\ D(x,y)\frac{\partial u}{\partial n} - \mathbf{q}\cdot\mathbf{n}u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $D(x,y) > 0$ is a given function of x and y, $\mathbf{q}(x,y)$ is a given vector-function of x and y, and $f(x,y)$ is a given function.

a Apply integration by parts to both divergence terms to derive the weak formulation. (2 pt)

b Derive the Galerkin Equations to the weak form in part a. (2 pt)

c We use linear triangular elements to solve the problem. All answers may be expressed in terms of $|\Delta_e|$, being twice the area of element e, and $\beta_i = \frac{\partial\phi_i}{\partial x}$ and $\gamma_i = \frac{\partial\phi_i}{\partial y}$.

i Compute the element matrix and element vector for an internal triangle. (3 pt)

ii Compute the element matrix and element vector for a boundary element. (1 pt)

You may use Newton-Cotes' approximation for numerical integration, which reads as

$$\int_e h(x,y)d\Omega = \frac{|\Delta_e|}{6}\sum_{p\in\{1,2,3\}} h(x_p,y_p), \text{ for triangle } e \text{ with vertices } (x_p,y_p), \quad p\in\{1,2,3\}.$$