Faculty of Electrical Engineering, Mathematics and Computer Science
Applied Finite Elements, Mastermath
EXAM APRIL 21, 2017: 13:30 – 16:30 O'Clock @ Educatorium Megaron, Utrecht University

1 Given Holand & Bell's Theorem for a line segment and for a triangle:

Theorem: Let e be a triangle in $\mathbb{R}^2$ with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3), and let $\lambda_1(x,y)$, $\lambda_2(x,y)$ and $\lambda_3(x,y)$ be linear on e, for which

$$\lambda_i(x_j,y_j) = \delta_{ij}, \qquad \text{where } \delta_{ij} \text{ represents the Kronecker Delta,}$$

and let m_1, m_2, $m_3 \in \mathbb{N} = \{0,1,2,\ldots\}$, then

$$\int_e \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3}\,d\Omega = \frac{|\Delta_e| m_1! m_2! m_3!}{(2+m_1+m_2+m_3)!}, \qquad \text{where } \frac{|\Delta_e|}{2} \text{ represents the area of triangle } e. \tag{1}$$

In this assignment, the λ-functions are always linear and always satisfy $\lambda_i(x_j,y_j) = \delta_{ij}$.

a Show that the Newton-Cotes numerical integration rule using linear functions over triangle e with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3) is given by

$$\int_e g(x,y)d\Omega \approx \frac{|\Delta_e|}{6}\sum_{p=1}^{3} g(x_p,y_p). \tag{2}$$

(1 pt)

b Next, we consider quadratic basisfunctions over triangle e with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3), and midpoints (x_4,y_4), (x_5,y_5) and (x_6,y_6) on the faces of e. For the quadratic functions, we use the following basisfunctions

$$\phi_i(x,y) = \lambda_i(2\lambda_i - 1), \text{ for } i \in \{1,2,3\},$$

and

$$\phi_4(x,y) = 4\lambda_1\lambda_2,\ \phi_5 = 4\lambda_2\lambda_3,\ \phi_6(x,y) = 4\lambda_3\lambda_1.$$

i Show that $\phi_i(x_j,x_j) = \delta_{ij}$ for $i,j \in \{1,\ldots,6\}$. (1 pt)

ii Show that the Newton-Cotes numerical integration using quadratic basis functions over triangle e is given by

$$\int_e g(x,y)d\Omega \approx \frac{|\Delta_e|}{6}\sum_{p=4}^{6} g(x_p,y_p). \tag{3}$$

(2 pt)

c Subsequently, we use the quadratic basis functions ϕ_i to calculate the entries of the element matrix. In this question, you may use $\frac{\partial\lambda_i}{\partial x} = \beta_i$ and $\frac{\partial\lambda_i}{\partial y} = \gamma_i$, and $|\Delta_e|$ representing twice the area of the triangular element e. Calculate the following integral

$$S_{11}^e = \int_e ||\nabla\phi_1||^2 d\Omega.$$

(3 pt)

2 Given the following functional, where $u(x,y)$ is subject to an essential boundary condition

$$J[u] = \int_\Omega \frac{1}{2}||\nabla u||^2 + \frac{1}{2}u^2 - uf(x,y)\,d\Omega,$$

$$u(x,y) = u_0(x,y), \qquad \text{on } \partial\Omega_1,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, where $\partial\Omega_1$ and $\partial\Omega_2$ are non-overlapping segments. We are interested in the minimiser for the above functional:

Find u, subject to $u = u_0(x,y)$ on $\partial\Omega_1$ such that $F(u) \leq F(v)$ for all v subject to $v = u_0(x,y)$ on $\partial\Omega_1$.

a Derive the Euler-Lagrange equation (PDE) for $u(x,y)$, and give the boundary condition(s) on $\partial\Omega$. (2 pt)

b Derive the Ritz equations. (1 pt)

c We approximate the solution to the minimisation problem by Ritz' Method. Give the element matrix based on linear triangular elements. You may use $|\Delta_e|$, being two times the area of element e, and $\frac{\partial \lambda_i}{\partial x} = \beta_i$ and $\frac{\partial \lambda_i}{\partial y} = \gamma_i$ for the basis functions. (2 pt)

3 We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ (bounded by $\partial\Omega$):

$$\begin{cases} \nabla \cdot \left(\dfrac{\nabla u}{\sqrt{1 + ||\nabla u||^2}} \right) = 0, & \text{in } \Omega, \\ \\ u = g(x,y), & \text{on } \partial\Omega, \end{cases} \tag{4}$$

a Derive the weak formulation in which the order of spatial derivatives is minimized. (2 pt)

b Derive the Galerkin Equations to the weak form in part a. (1 pt)

c We use Picard's Fixed Point Method to solve the nonlinear problem. Describe how you would approximate the solution using successive approximations. (2 pt)

d We use linear triangular elements to solve the problem. All answers may be expressed in terms of $|\Delta_e|$ being twice the area of element e, and in $\frac{\partial \lambda_i}{\partial x} = \beta_i$ and $\frac{\partial \lambda_i}{\partial y} = \gamma_i$.

   i Compute the element matrix and element vector for an internal triangle. (2 pt)

   ii Compute the element matrix and element vector for a boundary element. (1 pt)

$$\text{Exam Grade} = \frac{\text{Sum over all credits}}{2}.$$