Faculty of Electrical Engineering, Mathematics and Computer Science
Applied Finite Elements, Mastermath
EXAM MAY 22, 2017: 13:30 – 16:30 O'Clock @ BBG, room 169, Utrecht University

1 We consider bi-linear quadrilateral elements to solve a finite-element problem. In the element we define bi-linear basis functions $\phi_i(\mathbf{x})$ corresponding to the 4 vertices, with coordinates $\mathbf{x}_p$, $p \in \{1,2,3,4\}$, which are used to approximate the solution. The basis-functions are defined through $\phi_i(\mathbf{x}_j) = \delta_{ij}$.

a In this part of assignment 1, we use a quadrilateral element with vertices $\mathbf{x}_1 = (1,0)$, $\mathbf{x}_2 = (0,1)$, $\mathbf{x}_3 = (-1,0)$ and $\mathbf{x}_4 = (0,-1)$, and we define the bi-linear basis functions according to $\phi_i(x,y) = a_i + b_i x + c_i y + d_i xy$.

i Use the given coordinates for the vertices of the quadrilateral to show that for basis function ϕ_1, we have

$$\begin{aligned} a_1 + b_1 &= 1, \\ a_1 + c_1 &= 0, \\ a_1 - b_1 &= 0, \\ a_1 - c_1 &= 0. \end{aligned} \tag{1}$$

(1 pt)

ii Use the above result to demonstrate that the form $\phi_i(x,y) = a_i + b_i x + c_i y + d_i xy$ cannot be used. (2 pt)

b Since the form $\phi_i(x,y) = a_i + b_i x + c_i y + d_i xy$ cannot be used, we use an isoparametric transformation, which is defined by

$$(T): \mathbf{x}(s,t) = \mathbf{x}_1(1-s)(1-t) + \mathbf{x}_2 s(1-t) + \mathbf{x}_3 st + \mathbf{x}_4(1-s)t, \text{ with } s,t \in [0,1],$$

to map the quadrilateral onto a reference square.

i Consider the reference element $\tilde{e} = (0,1) \times (0,1)$, show that the Newton–Cotes Rule applied to a unit square, $\tilde{e}$, is given by

$$\int_{\tilde{e}} g(s,t) d\Omega = \frac{1}{4} \left(g(0,0) + g(1,0) + g(1,1) + g(0,1) \right). \tag{2}$$

(2 pt)

ii The transformation (T) can be rearranged into

$$\mathbf{x}(s,t) = \mathbf{x}_1 + (\mathbf{x}_2 - \mathbf{x}_1)s + (\mathbf{x}_4 - \mathbf{x}_1)t + (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4)st.$$

Show that the determinant of the Jacobian of the transformation is given by

$$\frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1 + A_x t)(y_4 - y_1 + A_y s) - (y_2 - y_1 + A_y t)(x_4 - x_1 + A_x s),$$

with $A_x = x_1 - x_2 + x_3 - x_4$ and $A_y = y_1 - y_2 + y_3 - y_4$, and compute the Jacobian on the four vertices that were given in assignment 1a. (1 pt)

iii Compute the Jacobian matrix of the inverse transformation. Evaluate the inverse Jacobian matrix on vertex $\mathbf{x}_1$ given in assignment 1a. (2 pt)

c What is the size of the element matrix for an internal quadrilateral element if a single partial differential equation is solved? Motivate the answer. (1 pt)

2 Given the following functional, where $u(x,y)$ is subject to an essential boundary condition

$$J[u] = \int_\Omega \sqrt{1 + ||\nabla u||^2} - u f(x,y) \, d\Omega,$$

$$u(x,y) = u_0(x,y), \qquad \text{on } \partial\Omega_1,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, where $\partial\Omega_1$ and $\partial\Omega_2$ are non-overlapping segments. We are interested in the minimiser for the above functional:

Find u, subject to $u = u_0(x,y)$ on $\partial\Omega_1$ such that $F(u) \leq F(v)$ for all v subject to $v = u_0(x,y)$ on $\partial\Omega_1$.

a Derive the Euler-Lagrange equation (PDE) for $u(x,y)$, and give the boundary condition(s) on $\partial\Omega$. (2 pt)

b Derive the Ritz equations. (1 pt)

We approximate the solution to the minimisation problem by Ritz' Method. Note that the system of equations is non-linear. In the Picard fixed point method, we use the solution from the previous iteration for the nonlinear part. Further we use piecewise linear basis functions

c Give the element matrix and vector, as well as the boundary element matrix and vector based on linear triangular elements. You may use $|\Delta_e|$, being two times the area of element e, and $\frac{\partial \lambda_i}{\partial x} = \beta_i$ and $\frac{\partial \lambda_i}{\partial y} = \gamma_i$ for the basis functions. Further, Newton-Cotes numerical integration should be used if it is not possible to evaluate the integrals exactly. (Newton-Cotes numerical integration reads as $\int_e g(x,y)d\Omega \approx \frac{|\Delta_e|}{6}\sum_{p\in\{1,2,3\}} g(x_p,y_p)$, where (x_p,y_p) represent the coordinates of the vertices of triangle e, and as $\int_{be} g(x,y)d\Gamma \approx \frac{|be|}{2}\sum_{p\in\{1,2\}} g(x_p,y_p)$, where (x_p,y_p) represent the coordinates of the vertices of line segment be.) (2 pt)

3 We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ (bounded by $\partial\Omega$):

$$\begin{cases} -\nabla \cdot (D(x,y)\nabla u) = f(x,y), & \text{in } \Omega, \\ \\ u = g(x,y), & \text{on } \partial\Omega, \end{cases} \tag{3}$$

a Derive the weak formulation in which the order of spatial derivatives is minimized. (2 pt)

b Derive the Galerkin Equations to the weak form in part a. (1 pt)

c We use linear triangular elements to solve the problem. All answers may be expressed in terms of $|\Delta_e|$ being twice the area of element e, and in $\frac{\partial \lambda_i}{\partial x} = \beta_i$ and $\frac{\partial \lambda_i}{\partial y} = \gamma_i$. Use the Newton-Cotes numerical integration method if a numerical integration method is needed.

i Compute the element matrix and element vector for an internal triangle. (2 pt)

ii Compute the element matrix and element vector for a boundary element. (1 pt)

$$\text{Exam Grade} = \frac{\text{Sum over all credits}}{2}.$$