Faculty of Electrical Engineering, Mathematics and Computer Science, TU Delft
Applied Finite Elements, Mastermath
EXAM MAY 03, 2021: 14:00 – 17:00

Read the following instructions carefully before beginning the exam:

- The Applied Finite Element Methods exam will be an open book exam. You are therefore allowed to use the course book: J. van Kan, A. Segal, F. Vermolen, Numerical Methods in Scientific Computing, DAP. You are not allowed to use any other sources of information (internet, other books, etc.).
- You are supposed to do the exam alone without help from other people or resources, except the course book.
- On the first page of your solutions, write your name, the name of your university, the following statement, and sign below it:

  This exam will be solely undertaken by myself, without any assistance from others, and without the use of other sources explicitly allowed on the exam.

- After you complete your exam, please take pictures of your answers, your student ID (or any other official ID if you don't have a student ID), and if applicable also your card allowing extra time.

  Please combine all pictures into one pdf file, other submissions will not be accepted.

- Email your submissions to afemastermath@gmail.com with the following subject:

  "AFE21, Exam, ⟨ name of your university ⟩"

  The PDF file name should be "⟨ your last name ⟩.pdf."

- You should submit your pdf file with exam results no later than 17.30 hours (for students allowed extra time 18:05 hours). Any submissions received after 17:30 hours (or 18.05 for extra time students) will be marked as late submissions and considered as not submitted.
- Around 17:30 hours an email will be sent asking a subset of the class to join a Zoom meeting for confirming your identity, asking possible follow-up questions, etc.
- After the exam is graded we might contact you to explain your exam answers to check the authenticity of your work.
- During the exam, questions for clarification can be asked through Skype: Deepesh.Toshniwal or jaapvegtjjw.
- Any significant irregularities will be reported to the Exam Committee.

1. We consider the following boundary value problem for $u = u(x,y)$ to be determined in $\Omega \subset \mathbb{R}^2$ bounded by $\partial\Omega$:

$$\nabla \cdot (\mathbf{v}u - D\nabla u) + \varepsilon f(u) = 0, \qquad \text{in } \Omega,$$

$$u = g_1, \qquad \text{at } \partial\Omega_1$$

$$\mathbf{v} \cdot \mathbf{n}u - D\frac{\partial u}{\partial n} = g_2, \qquad \text{at } \partial\Omega_2,$$

with $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ the boundary of Ω with exterior normal vector $\mathbf{n}$, $\mathbf{v}, f, g_1, g_2$ are given functions with $f(u)$ positive and bounded, $0 < \varepsilon << 1$ and $D = D(x,y) > 0$ in Ω.

   (a) Derive the weak formulation in which the order of spatial derivatives is minimized. (1 pt)

   (b) Derive the finite dimensional Galerkin Equations to the weak form. Use a Picard iteration to deal with the nonlinear term. (2 pt)

   (c) We use quadrilateral elements to solve this problem. Use an isoparametric transformation to transform the integrals in the Galerkin equations to a reference element and use a Newton-Cotes quadrature rule to evaluate the integrals.

      i. Let S^e and f^e be the element matrix and element vector, respectively, for an internal quadrilateral e. Compute the first entry of S^e and the first entry of f^e. (2 pt)

      ii. Let e be a boundary quadrilateral such that one of its edges is contained in $\partial\Omega_1$. Let S^e and f^e be the element matrix and element vector, respectively. Compute the first entry of S^e and the first entry of f^e. (3 pt)

   Note: in the above two subproblems, only a single entry is being asked for; you do not need to compute the entire matrix/vector.

2. Given the following minimal surface problem on $\Omega \subset \mathbb{R}^2$

$$\min_u \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, d\Omega$$

$$u = g \qquad \text{at } \partial\Omega$$

with g a given function.

   (a) Compute the Euler-Lagrange equations for this problem. (2 pt)

   (b) Use the Ritz method to derive a system of nonlinear equations for the unknowns u_j, when u is approximated with u_h as: (2 pt)

$$u_h(x,y) = \sum_{j=1}^{n+n_e} u_j \phi_j(x,y),$$

   with ϕ_j given basis functions and n_e the number of points where the essential boundary conditions are enforced.

3. We consider the following time-dependent problem for $u = u(x,t)$ to be determined in $\Omega = (0,L)$ for $t \in (0,T)$:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right) + u^2 = f, \qquad \text{in } \Omega,$$

$$u(0,t) = 0,$$
$$u(L,t) = 0,$$
$$u(x,0) = h(x),$$

with f and h given functions on Ω. Here f is a known function and A is a given positive constant.

(a) Derive the weak formulation in which the order of spatial derivatives is minimized. (1 pt)

(b) Derive the nonlinear Galerkin Equations to the weak form above. Use the implicit method of Euler (backward) to discretize in time and linear basis functions to discretize in space. (2 pt)

4. Let Ω be a non-empty region in $\mathbb{R}^d$. We consider the following weak form:

$$(W): \text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \nabla u \cdot \nabla \phi d\Omega = \int_\Omega \phi f d\Omega, \qquad \forall \phi \in H_0^1(\Omega),$$

where $H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$, and $H^1(\Omega) := \{u \in L^2(\Omega) : \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)\}$. Let $\Sigma_h(\Omega)$ be a finite dimensional subspace of $H_0^1(\Omega)$. We search the finite element approximation of u in $\Sigma_h(\Omega)$, given by

$$(W_h): \text{Find } u_h \in \Sigma_h(\Omega) \text{ such that } \int_\Omega \nabla u_h \cdot \nabla \phi_h d\Omega = \int_\Omega \phi_h f d\Omega, \qquad \forall \phi_h \in \Sigma_h(\Omega).$$

(a) Let $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$, and let $||.||$ represent the vector norm, show that (1 pt)

$$\int_\Omega ||\nabla u||^2 d\Omega = 0 \Longrightarrow u = 0, \text{ in } \Omega.$$

(b) Use the form (W_h) and the weak form (W) to show that (1 pt)

$$\int_\Omega \nabla(u - u_h) \cdot \nabla \phi_h d\Omega = 0, \qquad \phi_h \in \Sigma_h(\Omega). \tag{1}$$

(c) Let $\hat{u}_h \in \Sigma_h(\Omega)$, show that (1 pt)

$$0 \leq \int_\Omega ||\nabla(u - u_h)||^2 d\Omega = \int_\Omega \nabla(u - u_h) \cdot \nabla(u - \hat{u}_h) d\Omega, \qquad \hat{u}_h \in \Sigma_h(\Omega). \tag{2}$$

(d) Show that the above relation (2) implies (1 pt)

$$0 \leq \int_\Omega ||\nabla(u - u_h)||^2 d\Omega \leq \int_\Omega ||\nabla(u - \hat{u}_h)||^2 d\Omega, \qquad \hat{u}_h \in \Sigma_h(\Omega). \tag{3}$$

(e) Let $u_I \in \Sigma_h(\Omega)$ represent the interpolation of u onto $\Sigma_h(\Omega)$, further let h be a characteristic element diameter and p be the order of interpolation, then it is possible to prove that there exists a constant $K > 0$ that is independent of h such that

$$\left(\int_\Omega ||\nabla(u - u_I)||^2 d\Omega\right)^{\frac{1}{2}} \leq K h^p. \tag{4}$$

With $\varepsilon = u - u_h$ the finite element error, use the above interpolation estimate to derive an upper bound for the energy norm $:= \left(\int_\Omega \|\nabla\varepsilon\|^2\right)^{\frac{1}{2}}$ when using p-th order interpolatory basis functions. (1 pt)

$$\text{Written Exam Grade} = \max\left(\frac{\text{Sum over all credits}}{2}, 1\right).$$