Written Exam Numerical Bifurcation Analysis of Large Scale Systems
June 19, 2019

Duration: 3 hours.
In front of the questions one finds the weights used to determine the final mark of this written exam. The mark is 1+9 * ("total score"/"sum of weights").

Problem 1

Consider the problem

$$u_t = -\alpha \sin(2u) + \beta u_{xx} + \gamma x(1-x)$$

on $[0,1]$ with boundary conditions $u(0,t) = 0$, $u_x(1,t) = 0$ and some initial condition.

a. [3] Transform the PDE of the previous part into a system of ODEs by a space discretization using central finite differences of second order on a mesh with equal mesh sizes. Don't forget the boundary conditions.

b. [2] Suppose $\beta = 1$ and $\gamma = 0$, derive the standard linear eigenvalue problem that has to be solved to find the α for which there is another steady solution in addition to the trivial steady solution of the equation of part a?

c. [4] How can the eigensolution found in part b be used to find an initial guess (including magnitude) on the branch of nontrivial steady solutions near a bifurcation point?

d. [3] Suppose $\beta = 1$ and $\gamma = 0.1$. Explain at least two ways of getting on the so-called isolated branch of solutions where α is the continuation parameter.

e. [4] For the eigenvalue problem in b we can use the Lanczos method, why? Also describe this method and discuss its advantage with respect to the Arnoldi method.

f. [3] How can one speed up the determination of the critical eigenvalues?

g. [4] Derive the eigenvalue problem for the linear stability study of the nontrivial solution.

h. [3] For the natural continuation of the stable nontrivial solution one has to solve linear systems in the Newton process. Explain what kind of linear systems can be solved by the CG and GMRES method, respectively. Can we employ the CG method in each step of the Newton iteration?

Exam continues on other side!

Problem 2

The Marangoni effect is the generation of the surface tension gradient and hence a surface shear stress due to a temperature gradient. The Rayleigh-Bénard-Marangoni convection problem concerns the flow of a layer of liquid that is heated from below. To compute the possible convection patterns in a finite size two-dimensional box, you (as a student) aim to use continuation techniques on the following non-dimensional equations:

$$Pr^{-1}\left[\frac{\partial u}{\partial t} + \mathbf{v}.\nabla u\right] = -\frac{\partial p}{\partial x} + \nabla^2 u \tag{1a}$$

$$Pr^{-1}\left[\frac{\partial w}{\partial t} + \mathbf{v}.\nabla w\right] = -\frac{\partial p}{\partial z} + \nabla^2 w + RaT \tag{1b}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{1c}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \nabla^2 T \tag{1d}$$

where $\mathbf{v} = (u, w)$ is the velocity field, p is the pressure and T is the temperature. The boundary conditions are

$$z = 1 \; : \; \frac{\partial u}{\partial z} = Ma\frac{\partial T}{\partial x}; w = 0; \frac{\partial T}{\partial z} = -Bi\ T \tag{2a}$$

$$z = 0 \; : \; T = 1 \; ; \; u = w = 0 \tag{2b}$$

$$x = 0, A_x \; : \; u = w = \frac{\partial T}{\partial x} = 0 \tag{2c}$$

In the equations (1)-(2), the dimensionless parameters Pr (Prandtl), Ra (Rayleigh), A_x (Aspect ratio), Marangoni Ma and Bi (Biot) appear. Note that the Marangoni number appears only in the boundary conditions.

a. [5] How do you determine the starting solution of the continuation procedure? Provide a short explanation.

The discrete equations, with state vector $\mathbf{x}$, can be written as

$$\mathcal{M}\frac{d\mathbf{x}}{dt} = \Phi(\mathbf{x}, Ma) \tag{3}$$

where Ma was chosen as the control parameter.

b. [1] Provide explicit expressions of the elements of the matrix $\mathcal{M}$.

c. [4] Formulate the pseudo-arclength scheme to follow a branch of steady solutions of (3) in the parameter Ma from the starting solution.

d. [4] Describe briefly two different procedures, depending on the availability of a Jacobian matrix, to solve the numerical problem arising in c.

e. [3] Which bifurcation point is expected as the first one once Ma is increased for $Ra = 0$? Provide a short explanation.

f. [4] Shortly describe a procedure to compute the first bifurcation point, as indicated in e, on the branch originating from the starting solution when $Ra = 0$. N.B. the continuation is in Ma.