Written Exam Numerical Bifurcation Analysis of Large Scale Systems
June 16, 2021

Duration: 3 hours.
In front of the questions one finds the weights used to determine the final mark of this written exam. The mark is 1+9 * ("total score"/"sum of weights").

Problem 1

Consider the problem

$$u_t = (\alpha + 2i)u + \beta u_{xx} - |u|^2 u$$

on $[0, 1]$ with boundary conditions $u(0, t) = 0$, $u_x(1, t) = 0$ and some initial condition. Here, the coefficients α and β are real and $\beta \geq 0$.

a. [2] Define $u = v + iw$ where v and w are real functions and show that the equation is equivalent to the real system

$$\begin{aligned} v_t &= \alpha v - 2w + \beta v_{xx} - (v^2 + w^2)v, \\ w_t &= \alpha w + 2v + \beta w_{xx} - (v^2 + w^2)w. \end{aligned}$$

b. [3] Transform the system of the previous part into a system of ODEs by a space discretization using central finite differences of second order on a mesh with equal mesh sizes. Don't forget the boundary conditions!

c. [6] What is the trivial solution of the equation of part a that holds for any α and β? Suppose $\beta = 0$, derive the eigenvalue problem that follows from the linear stability study of the trivial steady solution of the equation of part a. Next determine the α for which another solution comes into existence. Show that it is a Hopf bifurcation and give the period of the solution that will emerge at the bifurcation.

d. [5] How is $[v, w] = \epsilon[\cos 2t, \sin 2t]$ related to the eigenvalue problem of the previous part? Substitute this expression into the system given in part a for $\beta = 0$ and derive from that how ϵ will behave as function of α for $\alpha > 0$. How can we use this result in a continuation?

e. [6] Next we set $\beta = 1$ in the equations in part a. To which eigenvalue problem does this lead for the linear stability analysis of the trivial solution?
Observe that $\sin(k\pi x)$, for integer k, is an eigenfunction of d^2/dx^2 with boundary conditions zero at $x = 0$ and 1. Use this to derive the 2x2 eigenvalue problem needed to study linear stability. For which α will the bifurcation point occur on the branch of trivial solutions?

f. [4] For the discrete case derived in b, sketch the structure of the Jacobian matrix as derived in part b if we order $v_1, \cdots, v_n, w_1, w_2, \cdots, w_n$, where n is the number of internal grid points used in b. Can one use the Lanczos method to solve the eigenvalue problem associated to the linear stability problem? Motivate your answer.

g. [4] Suppose we use the backward Euler method for the continuation of the periodic branch that will evolve after the bifurcation. Using the extended system approach, what will the correction equation in Newton's method look like?

Problem 2

A student wants to determine a bifurcation diagram of a model of the wind-forced Atlantic Ocean circulation, which is given by the following set of partial differential equations for the stream function ψ and vorticity ζ on the domain $[0,1]\times[0,1]$:

$$\begin{aligned}(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})(\zeta - F\psi)+\beta\frac{\partial \psi}{\partial x} &= \alpha\Big(\frac{\partial \tau^y}{\partial x}-\frac{\partial \tau^x}{\partial y}\Big)-r\zeta+\frac{1}{Re}\nabla^2\zeta,\\ \zeta &= \nabla^2\psi,\end{aligned}$$

where the horizontal velocities are given by $u=-\partial\psi/\partial y$ and $v=\partial\psi/\partial x$.

Kinematic conditions on all lateral boundaries are $\psi=0$ and slip conditions are assumed. Hence the boundary conditions are given by

$$\begin{aligned} x=0,1 &: \psi=0,\zeta=0,\\ y=0,1 &: \psi=0,\zeta=0.\end{aligned}$$

The wind-stress forcing is prescribed as

$$\begin{aligned}\tau^x(x,y) &= -\frac{1}{2\pi}\cos 2\pi y,\\ \tau^y(x,y) &= 0.\end{aligned}$$

Suppose these equations are discretized on an equidistant grid $(x_i,y_j)=(i/N,j/M)$ for $i=0,\ldots,N$ and $j=0,\ldots,M$. After discretization, they can be written as

$$M\frac{d\mathbf{x}}{dt}=\Phi(\mathbf{x},Re).$$

a. [5] Give the dimension of the state vector $\mathbf{x}$ and provide an explicit expression of the matrix M.

The student fixes the values of the parameters F, r and α and uses Re as a control parameter.

The student wants to determine a branch of steady solutions in Re, starting e.g. at $Re=10$ and will use pseudo-arclength continuation for these computations.

b. [6] Advise the student on how to compute a starting steady solution for $Re=10$ on this branch.

c. [5] Argue why the first bifurcation on this branch is of pitchfork type.

d. [5] Advise the student on two different ways to detect the value of Re at this pitchfork bifurcation using pseudo-arclength continuation.

It turns out that the value of Re at the pitchfork bifurcation is $Re_P\sim 29.5$ and the bifurcation is supercritical. From here, two branches of stable steady solutions appear.

e. [4] Advise the student how to efficiently detect Hopf bifurcations on these two branches. Why will the values of Re, say Re_H, be equal on both branches?

It turns out that the value of Re at the first Hopf bifurcation is $Re_H \sim 61$ and also this bifurcation is supercritical.

f. [5] Write down the pseudo-arclength continuation scheme to determine a branch of periodic solutions from this Hopf bifurcation and also describe how the starting solution on this branch can be calculated.