Written Exam Numerical Bifurcation Analysis of Large Scale Systems
June 7, 2017

Duration: 3 hours.

In front of the questions one finds the weights used to determine the final mark.

Problem 1

Consider the problem

$$u_t = -\alpha \cos(2u) + \beta u_{xx} + \gamma x(1-x)$$

on $[0, 1]$ with boundary conditions $u(0,t) = 3\pi/4$, $u_x(1,t) = 0$ and some initial condition.

a. [3] If $\beta = \gamma = 0$ and $\alpha > 0$ we find for every $x \in (0,1)$ an ODE. What are the steady states of this ODE and what can be said about their respective stabilities? What happens if $\gamma \neq 0$ but small with respect to α?

b. [3] Which of the steady states found in the previous part is also a steady state for $\beta \neq 0$ and $\gamma = 0$. Show that for the perturbation v of this state the original equation turns over into

$$v_t = -2\alpha \sin(2v) + \beta v_{xx} + \gamma x(1-x)$$

with $v(0,t) = 0$, $v_x(1,t) = 0$. What is the numerical advantage of using the equation for the perturbation instead of the original equation?

c. [3] Transform the PDE of the previous part into a system of ODEs by a space discretization using central finite differences of second order on a mesh with equal mesh sizes. Don't forget the boundary conditions.

Problem 2

The Rayleigh-Bénard convection problem concerns the flow of a layer of liquid that is heated from below. To compute the possible convection patterns in a finite size two-dimensional box, you (as a student) aim to use continuation techniques on the following non-dimensional equations:

$$Pr^{-1}\left[\frac{\partial u}{\partial t} + \mathbf{v}.\nabla u\right] = -\frac{\partial p}{\partial x} + \nabla^2 u \quad (1a)$$

$$Pr^{-1}\left[\frac{\partial w}{\partial t} + \mathbf{v}.\nabla w\right] = -\frac{\partial p}{\partial z} + \nabla^2 w + Ra\ T \quad (1b)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (1c)$$

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \nabla^2 T \quad (1d)$$

where $\mathbf{v} = (u, w)$ is the velocity field, p is the pressure and T is the temperature. The boundary conditions are

$$z = 1 \;:\; \frac{\partial u}{\partial z} = w = 0; \frac{\partial T}{\partial z} = -Bi\ T \quad (2a)$$

$$z = 0 \;:\; T = 1\ ;\ u = w = 0 \quad (2b)$$

$$x = 0, A_x \;:\; u = w = \frac{\partial T}{\partial x} = 0 \quad (2c)$$

In the equations (1)-(2), the dimensionless parameters Pr (Prandtl), Ra (Rayleigh), A_x (Aspect ratio) and Bi (Biot) appear.

a. [3] Independent from the discretization method used (e.g. finite difference, finite volume, spectral or finite element), which two properties of the continuous system have to be certainly inherited by the discretized system to compute the correct bifurcation diagrams? Provide a short explanation.

The discrete equations, with state vector $\mathbf{x}$, can be written as

$$\mathcal{M}\frac{d\mathbf{x}}{dt} = \Phi(\mathbf{x}, Ra) \tag{3}$$

where Ra was chosen as the control parameter.

b. [1] Provide explicit expressions of the elements of the matrix $\mathcal{M}$.

c. [4] Suppose we perform continuation of stationary states/equilibria in the Rayleigh number. Give the starting solution for $Ra = 0$. Starting from this, give a steady solution expressed in Ra for all $Ra > 0$? What does this solution describe physically?

d. [4] Formulate the pseudo-arclength scheme to follow a branch of steady solutions of (3) in the parameter Ra from a given starting solution.

e. [3] Shortly describe your procedure to compute the first bifurcation point on the branch described in part c. And describe your procedure to switch to the new branch bifurcating at that point from the branch.

Problem 3

Consider the problem

$$M\frac{d}{dt}u = F(u, \gamma).$$

where M is a symmetric positive definite mass matrix. Moreover, M, u en F are real and γ is a real parameter. During continuation of the steady solution of a PDE with real coefficients, for the stability study, the eigenvalue with the largest real part behaves in the neighborhood of $\gamma = 1$ like $\lambda = (\gamma - 1) + i$ where i is the imaginary unit.

a. [3] Derive the eigenvalue problem for the stability study.

b. [3] Prove that the complex conjugate of λ must also be an eigenvalue.

c. [3] What is the name of the occurring bifurcation at $\gamma = 1$? Describe the real solution splitting off at this bifurcation point.

d. [3] For the eigenvalue problem in (a) we cannot use Lanczos, why? However, one can use the Arnoldi method; describe and discuss this method.

e. [3] Describe what needs to be done to speed up the convergence to the mentioned eigenvalue.