Logic, a brief survey

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Contents

What is Logic?

Studies Correctness of Reasoning

Socrates is a man
All men are mortal
Therefore: Socrates is mortal

Also:

Socrates is a woman
All women are mortal
Therefore: Socrates is mortal

Why Logic? Formalization of Natural Language

Unambiguously

They saw the girl with binoculars

Consistency in reasoning

Niets gaat boven Groningen
Alles is beter dan niets
Dus: Alles is beter dan Groningen

Cruijff:

Van Italianen ken je niet verliezen,
maar ze kennen wel van je winnen!

Why Logic? Formalization of Natural Language

Example

Every man loves a woman

$\exists_{\text{Woman}} \forall_{\text{Man}}$: Man loves woman

Why Logic? Formalization of Natural Language

Example

Every man loves a woman

$\exists_{\text{Woman}} \forall_{\text{Man}}$: Man loves woman

Beyoncé

Why Logic? Formalization of Natural Language

Example

Every man loves a woman

$\exists_{\text{Woman}} \forall_{\text{Man}}$: Man loves woman

$\forall_{\text{Man}} \exists_{\text{Woman}}$: Man loves woman

Beyoncé

Why Logic? Formalization of Natural Language

Example

Every man loves a woman

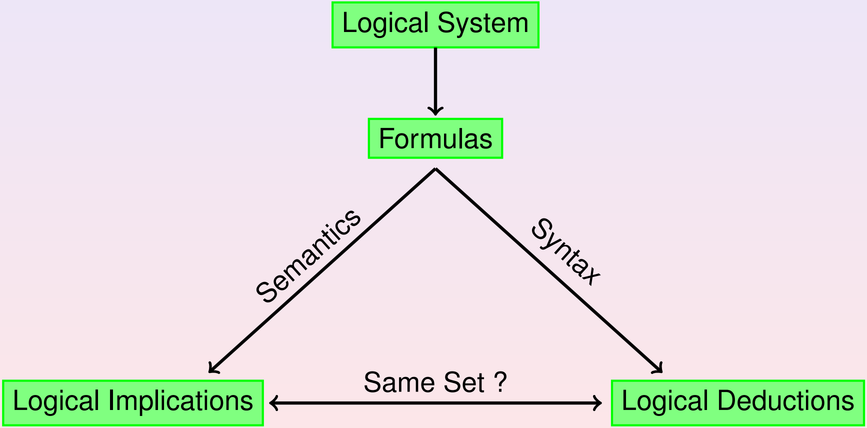
$\exists_{\text{Woman}} \forall_{\text{Man}}$: Man loves woman

Beyoncé

$\forall_{\text{Man}} \exists_{\text{Woman}}$: Man loves woman

My Mommy

Logical Systems: Syntax versus Semantics



Propositional Calculus (Zeroth-Order-Logic)

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Zeroth-Order Logical System

System $\mathcal{L} = (\mathcal{A}, \Omega, I, \mathcal{Z})$

- $\mathcal{A}$ is the alphabet, a countable set of proposition variables E.g. $\mathcal{A} = \{p_1, p_2, p_3, \ldots\}$.
- $\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_m$ is the set of operator symbols or logical connectives,
  where Ω_j denotes the set of operator symbols of arity j.
  Typically:
  $\Omega_0 = \{0, 1\}$ (the set of constant logical values) and
  $\Omega_1 = \{\neg\}; \quad \Omega_2 \subseteq \{\wedge, \vee, \rightarrow, \leftrightarrow, \underline{\vee}, |, \downarrow, \uparrow\}$.
- A set of brackets $\{(,), [,], \ldots\}$.
- I is a finite set of initial points or axioms.
- $\mathcal{Z}$ is a finite set of transformation rules or inference rules.

Syntax

The Language $\mathcal{L}$

The Language $\mathcal{L}$ is a set of formulas: the well-formed formulas or, wffs, which is recursively defined by the following rules:

1. Base: Any element of $\mathcal{A}$ is a formula of $\mathcal{L}$.
2. If $\phi_1, \phi_2, \ldots, \phi_j$ are formulas of $\mathcal{L}$, and $f \in \Omega_j$, then $(f(\phi_1, \phi_2, \ldots, \phi_j))$ is also a formula of $\mathcal{L}$,
   e.g. $(\wedge(\phi, \psi)) := (\phi \wedge \psi)$
3. Closed: Nothing else is a formula of $\mathcal{L}$.

Semantics (Interpretation)

Truth Tables (1)

ϕ	$\neg\phi$
0	1
1	0

Truth Tables (2)

ϕ	ψ	$\phi \wedge \psi$	$\phi \vee \psi$	$\phi \rightarrow \psi$	$\phi \leftrightarrow \psi$	$\phi \mid \psi$
0	0	0	0	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	0	1
1	1	1	1	1	1	0

Semantics: Truth Tables for Compound Statements

Truth Table for $\neg(p \rightarrow q) \vee (\neg q \wedge r)$ (*)

p	q	r	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$(\neg q \wedge r)$	*
0	0	0	1	0	1	0	0
0	0	1	1	0	1	1	1
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	1	1	0	1
1	0	1	0	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	0	0

Semantics: Tautologies and Logical Implications

Tautology

A Tautology is a formula ψ that is always true, regardless the truth values of its composing proposition variables. E.g, $p \vee \neg p$.
Notation: $\models \psi$

Logical Implication

A Logical Implication is a tautology of the form

$$(\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n) \to \psi.$$

$\phi_1, \ldots, \phi_n$ are the *premises*; ψ is the *conclusion*.
Notation: $\phi_1, \ldots, \phi_n \models \psi$; or: $\Sigma \models \psi$, where $\Sigma = \{\phi_1, \ldots, \phi_n\}$.

Logical Implication: Example

Truth Table for $((p \to q) \land (q \to r)) \to (p \to (q \land r))$ (*)

p	q	r	$p \to q\ (\phi_1)$	$q \to r\ (\phi_2)$	$\phi_1 \land \phi_2$	$p \to (q \land r)$	$*$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	0	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

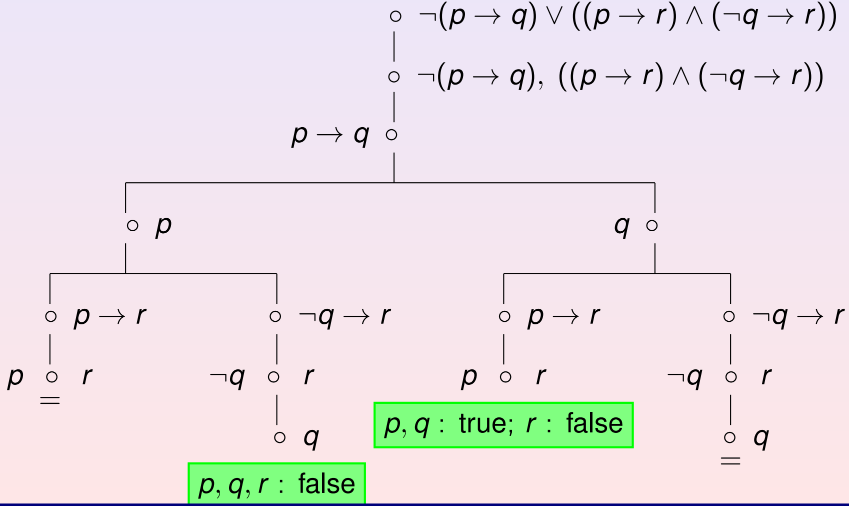
So the statement above is a logical implication $\phi_1, \phi_2 \models \psi$,
with premises $\phi_1 : p \to q$ and $\phi_2 : q \to r$
and conclusion $\psi : p \to (q \land r)$.

Semantics: Truth Table and Semantic Tableau

Truth Table for $\neg(p \to q) \vee ((p \to r) \wedge (\neg q \to r))$

p	q	r	$\neg(p \to q) \vee ((p \to r) \wedge (\neg q \to r))$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Semantic Tableau for $\neg(p \to q) \vee ((p \to r) \wedge (\neg q \to r))$

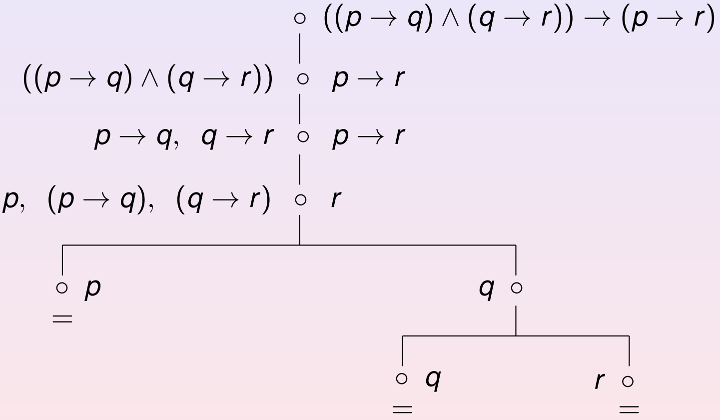


Semantics: Truth Table and Semantic Tableau

Truth Table for $((p \to q) \wedge (q \to r)) \to (p \to r)$ (*)

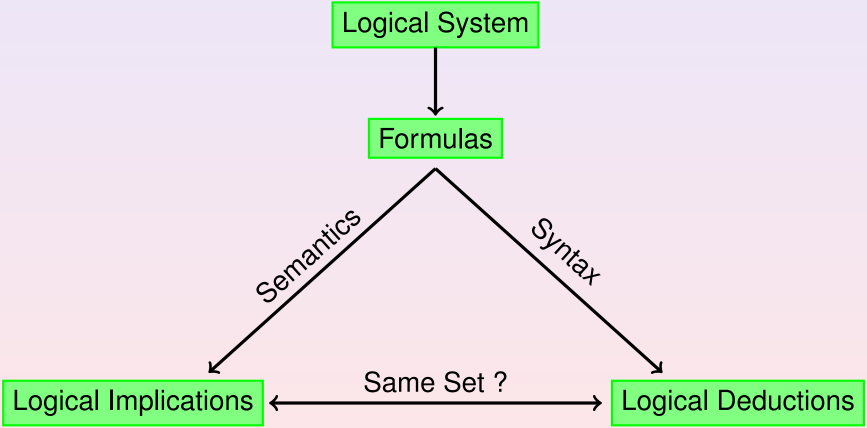
p	q	r	$p \to q$ (α_1)	$q \to r$ (α_2)	$\alpha_1 \wedge \alpha_2$	$p \to r$	$*$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Semantic Tableau for $((p \to q) \wedge (q \to r)) \to (p \to r)$



Tautology (Logical Implication)

Logical Systems: Syntax versus Semantics



Zeroth-Order Logical system

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Axiom Schemata for Propositional Calculus

The Standard Axioms (Lukasiewicz, 1917)

Ax1. $(\phi \to (\psi \to \phi))$

Ax2. $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

Ax3. $((\neg\phi \to \neg\psi) \to (\psi \to \phi))$

Rules of Inference: Modus Ponens

From the formulas ϕ and $(\phi \to \psi)$,
we may derive the new formula ψ.

Deductions (Proofs)

Deduction (Derivation)

A (direct) deduction of a conclusion ψ from a set of premises Σ is an ordered sequence of formulas such that each member of the sequence is either

(1) A premise or an axiom

(2) A formula derived from previous members of the sequence by one of the inference rules,

(3) The conclusion is the final step of the sequence.

Notation: $\Sigma \vdash \psi$

Theorems

If there exists a deduction of a formula ψ from the set of axioms I in a Logical System $\mathcal{L}$, then ψ is called a Theorem in $\mathcal{L}$.
Notation: $\vdash \psi$.

Deductions: Example

Law of the Syllogism: $(\phi \to \psi), (\psi \to \chi) \vdash (\phi \to \chi)$

Proof:

(1)	$(\psi \to \chi)$	Premise
(2)	$((\psi \to \chi) \to (\phi \to (\psi \to \chi)))$	Ax1
(3)	$(\phi \to (\psi \to \chi))$	(1), (2), MP
(4)	$((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$	Ax2
(5)	$((\phi \to \psi) \to (\phi \to \chi))$	(3), (4), MP
(6)	$(\phi \to \psi)$	Premise
(7)	$(\phi \to \chi)$	(5), (6), MP

Theorems: Example

Example in Standard System: $\vdash (\phi \to \phi)$

Proof:

(1) $(\phi \to (\phi \to \phi))$ Ax1

(2) $(\phi \to ((\phi \to \phi) \to \phi))$ Ax1

(3) $((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$ Ax2

(4) $(\phi \to (\phi \to \phi)) \to (\phi \to \phi))$ (2), (3), MP

(5) $(\phi \to \phi)$ (1), (4), MP

Theorems

Well known Theorems in Standard System

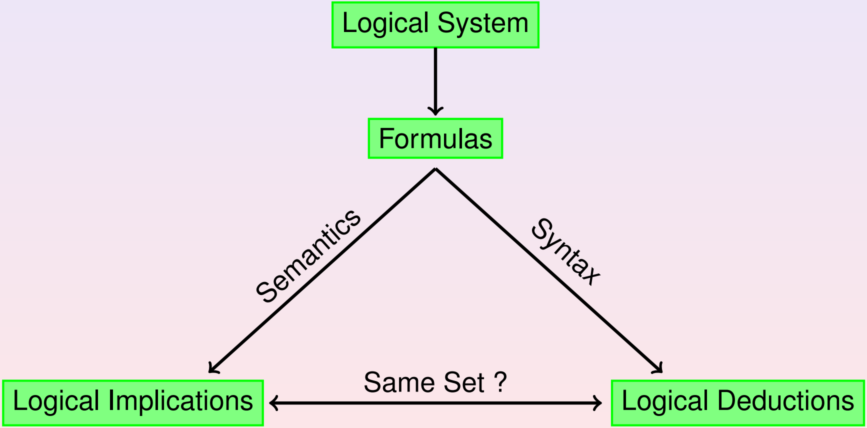
- $\vdash (\phi \to \phi)$
- $\vdash (\neg(\neg\phi) \to \phi)$
- $\vdash (\phi \to \neg(\neg\phi))$
- $\vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$
- $\vdash (\phi \vee \neg\phi)$
- $\vdash ((\phi \vee \phi) \to \phi)$
- $\vdash (((\phi \vee \psi) \vee \chi) \to (\phi \vee (\psi \vee \chi)))$
- $\vdash ((\phi \vee \psi) \to (\psi \vee \phi))$

Other Connectives expressed in terms of $\neg$ and $\rightarrow$

Definition of other connectives

- $(\phi \vee \psi) \equiv (\neg\phi \rightarrow \psi)$
- $(\phi \wedge \psi) \equiv \neg(\neg\phi \vee \neg\psi)$
- $(\phi \leftrightarrow \psi) \equiv ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$

Logical Systems: Syntax versus Semantics



Meta-Mathematical Properties of Propositional Calculus

Soundness Theorem

Let Σ be a set of formulas. Then

$$\Sigma \vdash \psi \quad \text{implies} \quad \Sigma \models \psi$$

If there exists a derivation of ψ from the Standard Axioms and the premisses in Σ, then $\Sigma \to \psi$ is a logical implication.

Completeness Theorem

Let Σ be a set of formulas. Then

$$\Sigma \models \psi \quad \text{implies} \quad \Sigma \vdash \psi$$

If $\Sigma \to \psi$ is a logical implication, then there exists a derivation of ψ from the Standard Axioms and the premisses in Σ.

Meta-Mathematical Properties of Propositional Calculus

Consistency Theorem

The set of Standard Axioms is Syntactically Consistent, i.e, there does not exist any formula ψ for which both

$$\vdash \psi \quad \text{and} \quad \vdash \neg\psi$$

Decidability Theorem

For each formula ψ there exist a finite, effective rote procedure to determine whether or not ψ is a theorem in $\mathcal{L}$.

Meta-Mathematical Properties of Propositional Calculus

Formula Induction

How to prove that a certain property $\mathcal{P}$ applies to all formulas?

(1) Show that $\mathcal{P}$ applies to each proposition variable $p \in \mathcal{A}$.

(2) For all $1 \leq j \leq m$ and all operator symbols $f \in \Omega_j$, show that, if $\mathcal{P}$ applies to the formulas $\phi_1, \phi_2, \ldots, \phi_j$, then it also applies to the formula $(f(\phi_1, \phi_2, \ldots, \phi_j))$.

Axiom Schemata for Propositional Calculus (1)

The Standard Axioms (Lukasiewicz, 1917)

Ax1. $(\phi \to (\psi \to \phi))$

Ax2. $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

Ax3. $((\neg\phi \to \neg\psi) \to (\psi \to \phi))$

Rules of Inference: Modus Ponens

ϕ and $(\phi \to \psi)$ imply ψ

Axiom Schemata for Propositional Calculus (2)

Principia Mathematica (Whitehead and Russell, 1910)

*PM*1	(*Tautology*) :	$((\phi \vee \phi) \rightarrow \phi)$
*PM*2	(*Addition*) :	$(\psi \rightarrow (\phi \vee \psi))$
*PM*3	(*Permutation*) :	$((\phi \vee \psi) \rightarrow (\psi \vee \phi))$
*PM*4	(*Associativity*) :	$((\phi \vee (\psi \vee \tau)) \rightarrow ((\psi \vee \phi) \vee \tau))$
*PM*5	(*Summation*) :	$((\psi \rightarrow \tau) \rightarrow ((\phi \vee \psi) \rightarrow (\phi \vee \tau)))$

Rules of Inference: Modus Ponens

ϕ and $(\phi \rightarrow \psi)$ imply ψ

Axiom Schemata for Propositional Calculus (3)

The Axiom Scheme of Meredith (1953)

$$(((((\phi \to \psi) \to (\neg\chi \to \neg\theta)) \to \chi) \to \tau) \to ((\tau \to \phi) \to (\theta \to \phi)))$$

Rules of Inference: Modus Ponens

ϕ and $(\phi \to \psi)$ imply ψ

The Axiom Scheme of Nicod (1917)

$$((\alpha \mid (\beta \mid \gamma)) \mid ((\delta \mid (\delta \mid \delta)) \mid ((\epsilon \mid \beta) \mid ((\alpha \mid \epsilon) \mid (\alpha \mid \epsilon)))))$$

Rules of Inference: Nicod's Modus Ponens

ϕ and $(\phi \mid (\psi \mid \chi))$ imply χ

Predicate Calculus (First-Order-Logic)

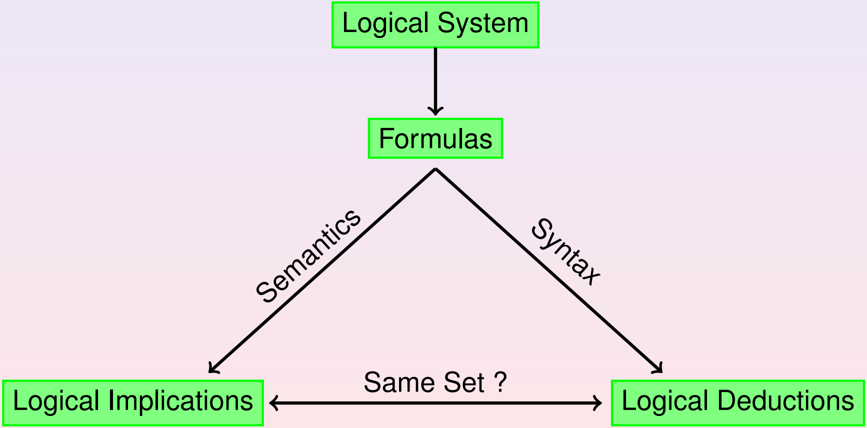
$$(\exists x\, P(x) \to \exists x\, Q(x)) \to (\exists x\, (P(x) \to Q(x)))$$

First-Order Logical System

Syntax of the System: $(\mathcal{C}, \mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{F}, \mathcal{I}, \mathcal{Z})$

- $\mathcal{C}$ is a set of constants (e.g. $\{0, 1, a, b, \ldots\}$)
- $\mathcal{V}$ is a set of variables (used as quantifier-variables): $\{x_1, x_2, \ldots\}$
- $\mathcal{L}$ is a set of logical symbols $\{\neg, \rightarrow, \vee, \wedge, \leftrightarrow, \forall, \exists, \ldots\}$
- $\mathcal{P}$ is a set of predicates. $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \ldots$, where $\mathcal{P}_k = \{P_1^k, P_2^k, \ldots\}$ denotes the set of predicates of arity k.
- $\mathcal{F}$ is a set of functions. $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \ldots$, where $\mathcal{F}_k = \{f_1^k, f_2^k, \ldots\}$ denotes the set of functions of arity k.
- A set of brackets $\{(,), [,], \ldots\}$.
- $\mathcal{I}$ is a set of initial points or axioms.
- $\mathcal{Z}$ is a set of transformation rules or inference rules.

Logical Systems: Syntax versus Semantics



Terms in Predicate Calculus

Terms

(1) Each constant $c \in \mathcal{C}$ and each variable $x \in \mathcal{V}$ is a term

(2) If $f^k \in \mathcal{F}_k$ is a function of arity k, and $t_1, \ldots, t_k$ are terms, then $f^k(t_1, \ldots, t_k)$ is also a term.

(3) Nothing else is a term.

Example

a, x_1, x_2,. . ., $f_1^2(a, x_9)$ and $f_2^2(x_5, x_9))$ are terms.

Formulas in Predicate Calculus

Formula

(1) If $t_1, \ldots, t_k$ are terms and $P \in \mathcal{P}_k$ is a predicate of arity k then $P(t_1, \ldots, t_k)$ is a formula (atomic formula).

(2) If ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \to \psi$, $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \leftrightarrow \psi$.

(3) If ϕ is a formula and $x \in \mathcal{V}$ is a variable, then $\forall x\, \phi$ and $\exists\, x\phi$ are formulas.

(4) Nothing else is a formula.

Example

$\forall x_1 \left[P_1^2(x_1, x_2) \wedge P_1^2(f_1^2(a, x_9), f_2^2(x_5, x_9))\right]$ is a formula.

Interpretation, Model, Distribution

Example

Consider the Syntax $(\mathcal{C}, \mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{F}, \mathcal{I}, \mathcal{Z})$, where
$\mathcal{C} = \{a\}$, $\mathcal{V} = \{x_1, x_2, \ldots\}$, $\mathcal{P} = \{P_1^2\}$ and $\mathcal{F} = \{f_1^2, f_2^2\}$.
Model $M = ((D, R, O), I)$ is given by
$D = \{1, 2, \ldots\}$, $R = \{<\}$ and $O = \{1, +, \cdot\}$
Interpretation I given by:
$I(P_1^2) = " < "$, $I(f_1^2) = " + "$, $I(f_2^2) = " \cdot "$ and $I(a) = 1$.
Distribution d is given by: $d(x_k) = k$ $(k = 1, 2, \ldots)$.
Then the formula $P_1^2(x_1, x_2) \wedge P_1^2(f_1^2(a, x_9), f_2^2(x_5, x_9))$
is interpreted as: "$1 < 2$ and $10 < 45$".

$M, d \models \phi$ $M \models \phi$ and $M \not\models \phi$

Example

Consider the Syntax $(\mathcal{C}, \mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{F}, \mathcal{I}, \mathcal{Z})$, where
$\mathcal{C} = \{a\}$, $\mathcal{V} = \{x_1, x_2, \ldots\}$, $\mathcal{P} = \{P_1^2\}$ and $\mathcal{F} = \{f_1^2, f_2^2\}$.
Consider a model Model $M = ((D, R, O), I)$, where
$D = \{1, 2, \ldots\}$, $R = \{<\}$ and $O = \{1, +, \cdot\}$
Interpretation I given by:
$I(P_1^2) = " < "$, $I(f_1^2) = " + "$, $I(f_2^2) = " \cdot "$ and $I(a) = 1$.
Let ϕ be the formula $\phi = P_1^2(x_1, x_2)$.
Then $M, d \models \phi$ for each distribution d with $d(x_1) < d(x_2)$.
Furthermore, $M \models \psi$, where ψ given by $P_1^2(x_1, f_1^2(x_1, x_2))$,
since $x_1 < x_1 + x_2$ in M.
If $D = \{0, 1, 2, \ldots\}$, then $M \not\models \psi$.

Logical Implications

Definitions

Let Σ be a set of formulas and ψ a formula.

Σ logically implies ψ (notation: $\Sigma \models \psi$) if the following implication holds for each model M:

If $\quad M \models \phi$ for each $\phi \in \Sigma, \quad$ then $\quad M \models \psi$.

Examples

- $\{\forall x\, P(x),\ \forall x(P(x) \to Q(x)\} \models \forall x\, Q(x)$.
- $\{\forall x\, P(x) \to \forall x\, Q(x)\} \not\models \forall x\, (P(x) \to Q(x))$.

Semantic Tableau for $(\forall x\, P(x) \to \forall x\, Q(x)) \to (\forall x\,(P(x) \to Q(x)))$

$\circ \; (\forall x\, P(x) \to \forall x\, Q(x)) \to (\forall x\,(P(x) \to Q(x)))$

$(\forall x\, P(x) \to \forall x\, Q(x)) \;\circ\; (\forall x\,(P(x) \to Q(x)))$

$\forall_{\mathbf{R}}: \;\circ\; P(d_1) \to Q(d_1) \qquad D = \{d_1\}$

$P(d_1) \;\circ\; Q(d_1)$

Left branch:

$\circ\; \forall x\, P(x)$

$\forall_{\mathbf{R}}: \;\circ\; P(d_2) \qquad D = \{d_1, d_2\}$

Right branch:

$\forall x\, Q(x) \;\circ$

$\forall_{\mathbf{L}}: \; Q(d_1) \;\circ$

$=$

Counterexample: $D = \{d_1, d_2\}$

$P(d_1)$: true; $P(d_2), Q(d_1)$: false

Semantic Tableau for $(\forall x\, P(x) \wedge \forall x(P(x) \rightarrow Q(x)) \rightarrow \forall x\, Q(x)$

$\circ \;\; (\forall x\, P(x) \wedge \forall x(P(x) \rightarrow Q(x)) \rightarrow \forall x\, Q(x)$

$\forall x\, P(x) \wedge \forall x(P(x) \rightarrow Q(x) \;\; \circ \;\; \forall x\, Q(x)$

$\forall x\, P(x),\; \forall x(P(x) \rightarrow Q(x) \;\; \circ \;\; \forall x\, Q(x)$

$\forall_{\mathbf{R}} : \;\; \circ \;\; Q(d_1) \qquad D = \{d_1\}$

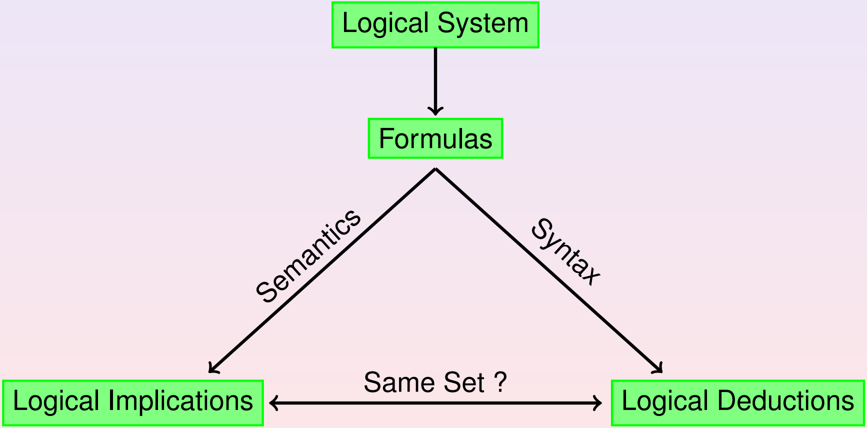
$\forall_{\mathbf{L}} : \;\; P(d_1) \;\; \circ$

$\forall_{\mathbf{L}} : \;\; P(d_1) \rightarrow Q(d_1) \;\; \circ$

$\circ \;\; P(d_1)$	$\forall_{\mathbf{L}} : \;\; Q(d_1) \;\; \circ$
$=$	$=$

Logical Implication

Logical Systems: Syntax versus Semantics



Axiom Schemata for Predicate Calculus

Ax. 1-3 for Prop. Calculus extended with Ax. 4-6 for Pred. Calculus

Ax1. $(\phi \rightarrow (\psi \rightarrow \phi))$

Ax2. $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$

Ax3. $((\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi))$

Ax4. $\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\, \phi \rightarrow \forall x\, \psi)$

Ax5. $\phi \rightarrow \forall x\, \phi$ provided that x is not free in ϕ

Ax6. $\forall x\, \phi \rightarrow [t/x]\, \phi$ provided that t is free for x in ϕ

Rules of Inference: Modus Ponens and Universal Generalization

MP ϕ and $(\phi \rightarrow \psi)$ imply ψ

UG $\Sigma \vdash \psi$ implies $\Sigma \vdash \forall x\, \psi$
provided that x is not free in any $\phi \in \Sigma$

Deductions in Predicate Calculus: Example

Deduction of $\forall x\, P(x),\ \forall x(P(x) \to Q(x)) \vdash \forall x\, Q(x)$

Proof:

(1) $\forall x(P(x) \to Q(x))$ Premise

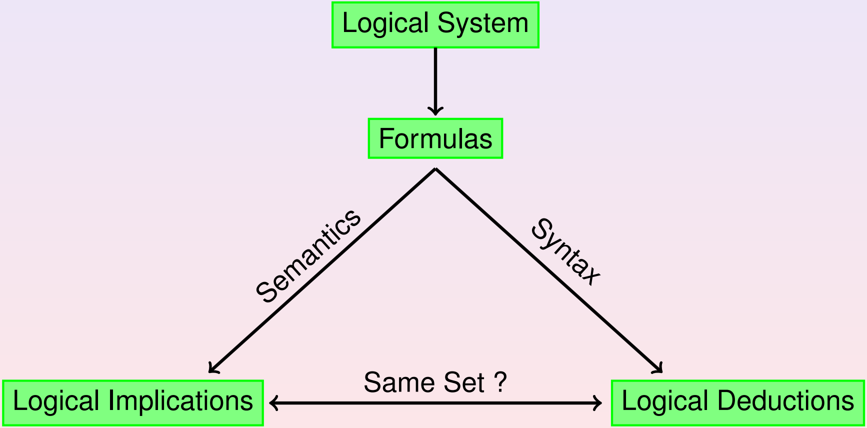
(2) $\forall x(P(x) \to Q(x)) \to (\forall x\, P(x) \to \forall x\, Q(x))$ Ax4

(3) $(\forall x\, P(x) \to \forall x\, Q(x))$ $(1), (2)$, MP

(4) $\forall x\, P(x)$ Premise

(5) $\forall x\, Q(x)$ $(4), (3)$, MP

Logical Systems: Syntax versus Semantics



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Soundness Theorem

Let Σ be a set of formulas. Then

$$\Sigma \vdash \psi \quad \text{implies} \quad \Sigma \models \psi$$

If there exists a derivation of ψ from the Axioms 1-6 and the premisses in Σ, then $\Sigma \models \psi$ is a logical implication.

Completeness Theorem (Gödel, 1930)

Let Σ be a set of formulas. Then

$$\Sigma \models \psi \quad \text{implies} \quad \Sigma \vdash \psi$$

If $\Sigma \models \psi$ is a logical implication, then there exists a derivation of ψ from the Axioms 1-6 and the premisses in Σ.

Meta-Mathematical Properties of Predicate Calculus

Consistency Theorem

The set of Axioms 1-6 is Syntactically Consistent, i.e,
there does not exist any formula ψ for which both

$$\vdash \psi \quad \text{and} \quad \vdash \neg\psi$$

Theorem: The System of Predicate Logic is not Decidable

There does not exist an effective rote decision procedure that determines whether arbitrary formulas are logically valid (provided that the system contains a predicate of arity at least 2).

Other Logical Systems: Restrictions

Monadic Predicate Logic

Predicate Logic where only unary predicates and no functions are allowed.
This system is decidable.

System of Universal Formulas

Only formulas of the form $\forall x_1 \forall x_2 \cdots \forall x_n\ [\psi]$ are allowed.

Other Logical Systems: Extensions

Second-Order-Logic

Also quantifications of predicates ($\forall P$) or functions ($\forall f$) are allowed. This system is not complete.

Higher-Order-Logic

Also predicates of predicates and functions of functions are allowed. E.g. differential-operators ∇f.

Many-Valued Logic; Modal Logic; Lambda Calculus

Other Logical Systems: Alternatives

Intuitionistic Logic

No Law of Excluded Middle.
So $p \vee \neg p$ is not a tautology, and $\neg\neg p$ does not necessarily imply p.
E.g: the non-constructive proof of the existence of $p, q \notin \mathbb{Q}$ with $p^q \in \mathbb{Q}$ is not accepted.

Theories and Axiomatic Systems

Theory of a Model

The Theory of model M is the set of all formulas that are true in M:
$Th(M) = \{\phi \mid M \models \phi\}$.

Axiomatic System for a Theory

A set of formulas Σ is an axiomatic system for theory $Th(M)$ if
$\phi \in Th(M)$ if and only if $\Sigma \models \phi$.

Euclids Axioms (Postulates) for Geometry

(1) For each two different points A and B, there exists a unique line passing through A and B.

(2) For each two different line segments AB and CD there exist a point E such that B is between A and E, and CD is congruent to BE.

(3) For each two different points O and A, there exists a circle with center O and radius A.

(4) All right angles are congruent.

(5) For each line ℓ and each point P not on ℓ, there exists a unique line m through P parallel to ℓ.

Tarski (1959) axiomatized Euclidean Geometry in first-order logic with 11 axioms, using a betweenness and congruence relation.

The Zermelo-Fraenkel Axioms for Set Theory (7 Axioms)

Let $\mathcal{C} = \{\varnothing\}$, $\mathcal{P} = \{P_1^2, P_2^2\}$ and $\mathcal{F} = \{f^1\}$.
Let $R = \{\in, \cup\}$ and $I(\varnothing) = "\varnothing"$, $I(P_1^2) = " \in "$, $I(P_2^2) = " \cup "$.

(1) $\forall X \forall Y (\forall u (u \in X \leftrightarrow u \in Y) \to X = Y)$
(Axiom of Extensionality)

(2) $\forall a \forall b \exists X \forall u (u \in X \leftrightarrow (u = a \vee u = b))$ (Axiom of Pairing)

(3) $\forall X \forall P \exists Y \forall u (u \in Y \leftrightarrow (u \in X \wedge P(u)))$ (Axiom of Subsets)

(4) $\forall X \exists Y \forall u (u \in Y \leftrightarrow \exists Z (u \in Z \wedge Z \in X))$
(Axiom of Unions)

(5) $\forall X \exists Y \forall Z (Z \in Y \leftrightarrow \forall u (u \in Z \to u \in X))$
(Axiom of the Power Set)

(6) $\exists S [\varnothing \in S \wedge (\forall x (x \in S \to (x \cup \{x\} \in S)))]$
(Axiom of Infinity)

(7) $\forall S [\exists x (x \in S) \to (\exists y (y \in S \wedge \neg \exists z (z \in y \wedge z \in S)))]$
(Axiom of Foundation)

Example: The Peano Arithmetic Axioms on $\mathbb{N}$

Let $\mathcal{C} = \{0\}$, $\mathcal{P} = \varnothing$ and $\mathcal{F} = \{f^1, f_1^2, f_2^2\}$.
Let $D = \mathbb{N}$, $R = \{0\}$, $O = \{S, +, \cdot\}$ and
$I(0) = 0$, $I(f^1) = S$ (S is the successor-operator),
$I(f_1^2) = " + "$ and $I(f_2^2) = " \cdot "$.

(PA1) $\forall x \, \neg(0 = S(x))$ (0 is not the successor of any number)

(PA2) $\forall x \, \forall y \, (S(x) = S(y) \rightarrow x = y)$ (S is an injective operation)

(PA3) $\forall x \, (x + 0 = x)$
$\forall x \, \forall y \, (x + S(y) = S(x + y))$
(recursion equations for addition)

(PA4) $\forall x \, (x \cdot 0 = 0)$
$\forall x \, \forall y \, (x \cdot S(y) = x \cdot y + x)$
(recursion equations for multiplication)

(PA5) $([0/x] \, \phi \wedge \forall x \, (\phi \rightarrow [S(x)/x] \, \phi)) \rightarrow \forall x \, \phi$
(induction principle)

Proof of $1 + 1 = 2 \quad (S0 + S0 = SS0)$

(1) $\forall x \forall y\, (x + Sy = S(x + y))$ PA3

(2) $S0 + S0 = S(S0 + 0)$ (1), Ax6 (2×) (x: $S0$; y: 0)

(3) $\forall x\, (x + 0 = x)$ PA3

(4) $S0 + 0 = S0$ (3), Ax6 (x: $S0$)

(5) $\forall x \forall y\, (x = y \rightarrow (S0 + S0 = Sx \rightarrow S0 + S0 = Sy))$
EQ3 (ϕ: $S0 + S0 = Sz$)

(6) $S0 + 0 = S0 \rightarrow (S0 + S0 = S(S0 + 0) \rightarrow S0 + S0 = SS0)$
(5), Ax6 (2×) (x: $S0 + 0$; y: $S0$)

(7) $S0 + S0 = S(S0 + 0) \rightarrow S0 + S0 = SS0$ (4),(6), MP

(8) $S0 + S0 = SS0$ (2),(8), MP

Proof of $\forall x\,(0 + x = x)$

(1) $\forall x\,(x + 0 = x)$ PA3

(2) $0 + 0 = 0$ (1), Ax6 (x: 0)

(3) $0 + n = n$ Premise

(4) $\forall x \forall y\,(x + Sy = S(x + y))$ PA3

(5) $0 + Sn = S(0 + n)$ Ax6 (2×) (x: 0; y: n)

(6) $\forall x \forall y\,(x = y \to (0 + Sn = Sx \to 0 + Sn = Sy))$
EQ3 (ϕ: $0 + Sn = Sz$)

(7) $0 + n = n \to (0 + Sn = S(0 + n) \to 0 + Sn = Sn)$
(5), Ax6 (2×) (x: $0 + n$; y: n)

(8) $0 + Sn = S(0 + n) \to 0 + Sn = Sn$ (3),(7), MP

(9) $0 + Sn = Sn$ (5),(8), MP

(10) $0 + n = n \to 0 + Sn = Sn$
(3),(9), Tarski (withdraws premise (3))

(11) $\forall x\,(0 + x = x \to 0 + Sx = Sx)$ (10), UG (no premises left)

(12) $\forall x\,(0 + x = x)$ (2),(11), PA5 (ϕ: $0 + x = x$)

The Formalization of Mathematics

David Hilbert (1862-1943)
"Wir müssen wissen -
Wir werden wissen! "

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Kurt Gödel (1906-1978)
"This blindness of logicians is indeed surprising"

Gödels Incompleteness Theorems (1931)

First Incompleteness Theorem

Any effectively generated (recursively enumerable) theory capable of expressing elementary arithmetic, cannot be both consistent and complete.
In particular, for any such theory, there exists a true statement that is not provable in that theory.

Second Incompleteness Theorem

Any consistent effectively generated theory capable of expressing elementary arithmetic, cannot prove its own consistency.
In particular, if such theory can prove its own consistency, it is inconsistent.

An unprovable theorem in Peano Arithmetic

The Paris-Harrington Theorem (1977)

The strengthened finite Ramsey Theorem is unprovable in Peano Arithmetic (but provable in second-order arithmetic).

The strengthened finite Ramsey Theorem

For any $n, k, m \in \mathbb{N}$ there exist $N \in \mathbb{N}$ such that if we color each n-element subset of $S = \{1, 2, \ldots, N\}$ with one of k colors, then we can find $Y \subseteq S$, with $|Y| \geq m$ and $|Y| \geq \min\{y \mid y \in Y\}$, such that all n-element subsets of Y have the same color.